CONDITIONS FOR FORMING THERMAL STRUCTURES IN A METALLIC CONDUCTOR HEATED BY AN ELECTRIC CURRENT

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A nonlinear metallic conductor with a temperature-dependent electric conductivity contains an inhomogeneity. Under certain conditions, under the action of Joule heat release in this medium a structure is formed as the inhomogeneity intergrows across the current lines.

The present work is an extension of studies [1, 2] that were carried out to investigate the interaction of thermal and electric fields in a heterogeneous nonlinear metallic conductor heated by an electric current that passes through it. Its main goal is to determine the most general conditions under which structures are formed in this system.

First, we recall the main results obtained earlier. The heating of an infinite metallic conductor with an inhomogeneity at the center described by the function $\Omega(r, z)$ by an electric current with the density $j_0(t)$ at infinity is studied. The current is directed along the z axis of the cylindrical coordinate system (r, z). The effect of the change in the current with time on the thermal processes in the vicinity of the inhomogeneity is analyzed in [2]. Therefore, for simplicity it will be assumed in what follows that $j_0(t) = j_0 =$ const. The equations describing the changes in the temperature T and the electric potential Φ in a conductor with the electric conductivity $\sigma = [1 + a\Omega(r, z)]^{-1}T^{-1}$ have the form $\nabla^2\Phi = -\nabla\Phi\cdot\nabla\sigma/\sigma$, $\partial T/\partial t = \nabla^2 T + \delta_0\sigma\nabla\Phi\cdot\nabla\Phi$, where the coordinates r and z are written in units of the characteristic dimension of the inhomogeneity R and the time t and the potential Φ are expressed in units of R^2/χ and $R^2 j_0/\sigma_0$, respectively; $T = 1 + \alpha(T_a - T_0)$; σ is the electric conductivity of the medium referred to the initial electric conductivity σ_0 at infinity; $a = (1/\sigma - 1)$ at $t = 0$, $r = z = 0$ is the parameter of the inhomogeneity; $\delta_0 = R^2\alpha j_0^2/(\chi c\gamma\sigma_0)$. Let $T = \exp(\delta_0 t)[1 + aT_1 + O(a^2)]$, $\Phi = \exp(\delta_0 t)[-z + a\Phi_1 + O(a^2)]$. Then

$$\nabla^2\Phi_1 = -\frac{\partial T_1}{\partial z} - \frac{\partial\Omega}{\partial z},$$

$$\frac{\partial T_1}{\partial t} = \nabla^2 T_1 - \delta_0\left(2\frac{\partial\Phi_1}{\partial z} + 2T_1 + \Omega\right), \quad T_1|_{t=0} = 0. \tag{1}$$

In [1, 2] problem (1) was studied when the inhomogeneity had the specific spherically symmetric shape $\Omega(r, z) = \exp(-r^2 - z^2)$. In this case the time dependence of the temperature at the center is

$$T_1(0, 0, t) = \frac{1}{2}\int_0^t \frac{d\tau}{\tau(1 + 4\tau)^{3/2}}\left[1 - (1 + 2\delta_0\tau)\frac{F(\sqrt{2\delta_0\tau})}{\sqrt{2\delta_0\tau}}\right], \tag{2}$$

where $F(x) = \exp(-x^2)\int_0^x \exp y^2 dy$ is the Dawson integral [3]. Function (2) reaches a minimum at

$$t = t_m \approx 1.703/(2\delta_0), \tag{3}$$

and at $t \to \infty$ it tends to the limit

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$$T_1(0, 0) = [\ln(8\delta_0) + C - 6]/4, \tag{4}$$

where $C = 0.5772$ ... is the Euler constant. The specific electric resistance ρ_1 and the temperature T_1 are connected by the relation

$$\rho_1 = T_1 + \Omega. \tag{5}$$

It the present study the solution of problem (1) is investigated for a wide class of functions describing the shape of the inhomogeneity. In particular, the effect of this shape on formation of structures is studied. It is required that the following conditions be fulfilled:

1. The function $\Omega(r, z)$ is continuously differentiable at least twice with respect to r and z at any point of the plane (r, z).

2. At $r = z = 0$ the function $\Omega(r, z)$ reaches the maximum, value $\Omega(0, 0) = 1$, and at $r \neq 0$, $z \neq 0$ it decreases monotonically so that $\lim\limits_{r\to\infty} \Omega(r, z) = \lim\limits_{z\to\infty} \Omega(r, z) = 0$

$$\lim_{r\to\infty} \frac{\partial\Omega}{\partial r} = \lim_{z\to\infty} \frac{\partial\Omega}{\partial z} = 0, \quad 0 < \Omega(r, z) < 1.$$

3. For any $z \in (-\infty, \infty)$ the functions

$$\int_0^\infty r\,|\ln r\,\Omega|\,dr, \quad \int_0^\infty r\left|\ln r\,\frac{\partial\Omega}{\partial z}\right| dr, \quad \int_0^\infty r\left|\ln r\,\frac{\partial^2\Omega}{\partial z^2}\right| dr. \tag{6}$$

exist.

4. At $z \to \infty$, $\Omega(r, z) = o(z^{-2})$.

In what follows, in order not to digress from the main subject, consequences of the above conditions will be written. From 2 and 3 it follows that the following integrals are convergent:

$$\omega(z) = \int_0^\infty x\Omega(x, z)\,dx, \quad a_0(z) = \int_0^\infty x\frac{\partial\Omega(x, z)}{\partial z}\,dx, \quad \int_0^\infty x\frac{\partial^2\Omega(x, z)}{\partial z^2}\,dx, \tag{7}$$

In this case, both (6) and (7) converge uniformly in z. Because of this it is possible to pass to the limit in z under the integral sign in (6) and (7) and to integrate these expressions. From the convergence of (7) it follows that at $r \to \infty$

$$\Omega(r, z) = o(r^{-2}), \quad \frac{\partial\Omega}{\partial z} = o(r^{-2}), \quad \frac{\partial^2\Omega}{\partial z^2} = o(r^{-2}). \tag{8}$$

Omitting the proof of these consequences, we point out that they can be derived from well-known theorems on integration of functions that depend on a parameter (for example, [4, 5]).

To find steady-state distributions $T_1(r, z)$, it will be assumed in (1) that $\partial T_1/\partial t = 0$. It is shown in Appendix 1 that at $\delta_0 >> 1$ the solution of (1) that is bounded at any point of the plane (r, z) has the form

$$T_1(r, z) = \frac{1}{2}\left[\int_r^\infty x\ln x\frac{\partial^2\Omega(x, z)}{\partial z^2}\,dx - \ln r\int_r^\infty x\frac{\partial^2\Omega(x, z)}{\partial z^2}\,dx\right] -$$

$$-\frac{\Omega(r, z)}{2} - \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^\infty \exp(ikz)\,ika(k)\,K_0\left(\frac{k^2}{\sqrt{2}}\xi\right) dk, \tag{9}$$

and the temperature distribution along the z axis is

$$T_1(0, z) = \frac{1}{2} \int_0^\infty x \ln x \frac{\partial^2 \Omega(x, z)}{\partial z^2} dx - \frac{\ln \sqrt{8\delta_0} - C}{2} \frac{da_0(z)}{dz} -$$

$$- \frac{\Omega(0, z)}{2} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(ikz)\, ika(k) \ln k dk , \tag{10}$$

where $a(k)$ is the Fourier transform of the function $a_0(z)$ from Eq. (3):

$$a(k) = ik \int_0^\infty x\Omega(x, k) dx = ik\omega(k), \quad \Omega(x, k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-ikz)\, \Omega(x, z) dz . \tag{11}$$

The temperature distribution along the r axis is found from (9) at $z = 0$. With account for (11) it can be written as follows:

$$T_1(r, 0) = \frac{1}{2} \left[\int_r^\infty x \ln x \frac{\partial^2 \Omega(x, 0)}{\partial z^2} dx - \ln r \int_r^\infty x \frac{\partial^2 \Omega(x, 0)}{\partial z^2} dx \right] -$$

$$- \frac{\Omega(r, 0)}{2} + \frac{1}{\sqrt{2\pi}} \int_0^\infty k^2 \omega(k) K_0 \left(\frac{k^2}{\sqrt{2}} \xi \right) dk . \tag{12}$$

The estimation of the integrals in (10) and (12) carried out in Appendix 2 makes it possible to determine the order of magnitude of the functions $T_1(0, z)$ and $T_1(r, 0)$ at large values of the cooordinates. Let $\Omega(0, z) = O(z^{-2-\varepsilon})$, where $\varepsilon > 0$. At $0 < \varepsilon < 1$, $T_1(0, z) = O(z^{-2-\varepsilon})$ irrespective of a change in the function $\Omega(r, z)$.

Now let $\Omega(r, 0) = O(r^{-2-\varepsilon})$. Also let $\partial^2\Omega/\partial z^2 = O(r^{-2-\varepsilon})$, $\varepsilon > 0$. Then, if $0 < \varepsilon < 3/2$, then $T_1(r, 0) = O(r^{-\varepsilon})$, that is, the temperature distribution over r is determined by the rate of decrease of the function $\Omega(r, 0)$. If $\varepsilon \geq 3/2$ and also if $\partial^2\Omega/\partial z^2 = O(r^{-4-\varepsilon})$ at any $\varepsilon > 0$, then $T_1(r, 0) = O(\xi^{-3/2}) = O(\delta_0^{3/4} r^{-3/2})$ irrespective of the specification of the function $\Omega(r, 0)$. Thus, if at large distances from the center $\Omega(r, 0) = O(r^{-m})$, $m \geq 7/2$, and $\Omega(0, z) = O(z^{-n})$, $n \geq 3$, then the change in the temperature in the plane (r, z) is independent of the initial shape of the inhomogeneity. A characteristic feature of the thermal structures is that at $t \to \infty$ the solution "forgets" the initial conditions, and the form of the structure becomes universal. Therefore, the exponents $m = 3/2$ and $n = 3$ are critical parameters.

For a qualitative description of the geometry of the thermal structures, $z = 0$ is assumed in Eq. (10) and the temperature at the center is determined:

$$T_1(0, 0) = \frac{1}{4} \left| - \frac{da_0(0)}{dz} \right| \ln \delta_0 - \theta_0/2 ,$$

where

$$\theta_0 = 1 - \frac{\ln 8 - C}{2} \left| - \frac{da_0(0)}{dz} \right| -$$

$$- \int_0^\infty x \ln x \frac{\partial^2 \Omega(x, 0)}{dz^2} dx - \sqrt{\left(\frac{2}{\pi}\right)} \int_{-\infty}^{\infty} ika(k) \ln k dk .$$

At $\delta_0 = \delta_{0\,cr} = \exp(2\theta_0 / |da_0(0)/dz|)$, $T_1(0, 0) = 0$. This value of the parameter δ_0 separates two regimes. At $\delta_0 > \delta_{0\,cr}$, $T_1(r, 0) > 0$. Bearing in mind that the function $T_1(r, 0)$ decreases more slowly at $r >> 1$ than $T_1(0, z)$ at $z >> 1$, it can be concluded that in this regime a distinctive effect appears: the inhomogeneity intergrows across the current lines. In this case the distribution of the temperature and, consequently, of specific resistance ρ_1 of (5) takes the form of a "ridge" that extends along the r axis. It should be noted that the larger δ_0, the higher the

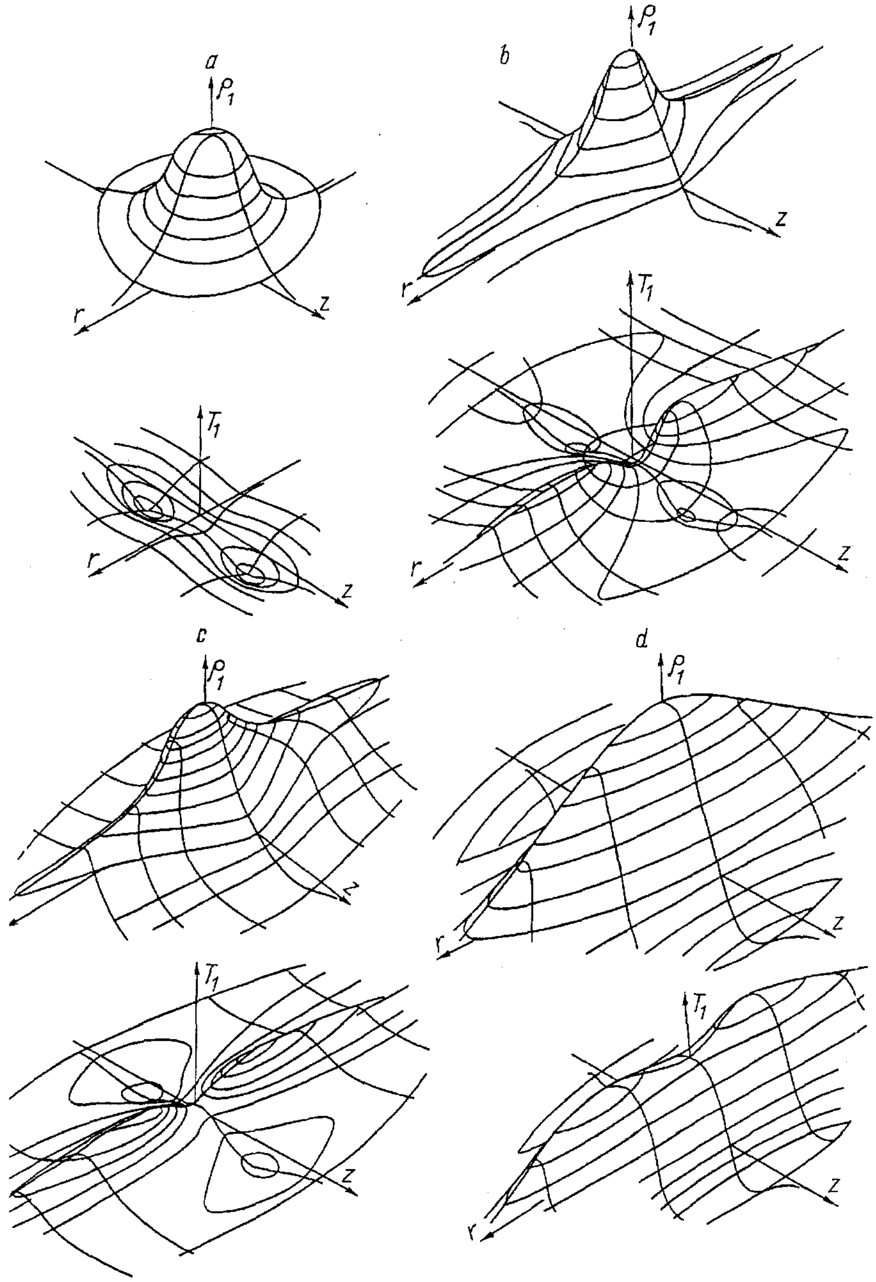

Fig. 1. Sketches of the specific resistance $\rho_1 = \Omega$ and the temperature T_1 at $\delta_0 = 0.1$ (a), 20 (b), 40 (c), and 10^4 (d).

temperature and the smaller the effect of the function $\Omega(r, z)$ on the form of the "ridge"of the specific resistance. Therefore, apart from the thermal structure, an electric structure appears in the system.

If $\delta_0 < \delta_{0\ cr}$, the temperature at the center becomes negative: destruction of both the thermal and the electric structure is observed. The form of the specific resistance becomes similar to that of the initial inhomogeneity $\Omega(r, z)$, and the temperature distribution takes the form of a "ravine" along the current lines rather than the form of a "ridge" extended across the current lines.

Figure 1 shows results of the numerical solution of problem (1) for a spherically symmetric inhomogeneity at different values of the parameter δ_0. As follows from (4), $\delta_{0\ cr} \approx 28$ for this inhomogeneity (In [1] $\delta_{0\ cr}$ was calculated with an error).

Hitherto, the distributions of temperature and specific resistance were considered in the steady-state regime ($t \to \infty$). The problem of estimating the time of development of the steady-state temperature regime (the time of structure formation) arises. For simplicity, spherical symmetry of the inhomogeneity is assumed. The quantity defined by

$$t_0 = t_m + (T_{10} - T_{1m})/(\partial T_1/\partial t)_{\max}, \qquad (13)$$

is called the time of development of the steady-state temperature regime. In (13) t_m and $T_{10} = T_1(0, 0)$ are prescribed by expressions (3), (4); T_{1m} is the value of function (2) at $t = t_m$; $(\partial T_1/\partial t)_{\max}$ is the maximum rate of change of function (2). This formula corresponds to the course of the change of temperature with time [1].

The derivative of (2) with respect to time is

$$\frac{\partial T_1}{\partial t} = \frac{1}{2}\,\frac{1}{t\,(1 + 4t)^{3/2}}\left[1 - (1 + 2\delta_0 t)\,\frac{F(\sqrt{2\delta_0 t})}{\sqrt{2\delta_0 t}}\right]. \qquad (14)$$

Let $\delta_0 >> 1$. In (14) the change in t will be neglected compared to $\delta_0(t)$, and from the condition $d^2T_1/dt^2 = 0$ the equation $x_m(1 + x_m^2) = (1 + x_m^2 + 2x_m^4)F(x_m)$ is obtained, where $x_m = \sqrt{2\delta_0 t_{\max}}$. Hence $t_{\max} \approx x_m^2/(2\delta_0)$ is found, and substituting it into (14), $(dT_1/dt)_{\max} = \delta_0 B = \delta_0 \cdot \mathrm{const}$ is obtained. The temperature T_{1m} is found by substitution of (3) into (2):

$$T_{1m} \approx \frac{1}{2\sqrt{2\delta_0 t_m}}\,\frac{1}{(1 + 4t_m)^{3/2}} \times$$

$$\times \int\limits_0^{\sqrt{2\delta_0 t_m}} dx \left[1 - (1 + x^2)\,\frac{F(x)}{x}\right] \approx A = \mathrm{const}.$$

Then, with account for (4) $t_0 \approx 1.703/(2\delta_0) + (\ln \delta_0 - A)/(B\delta_0)$. Thus, at $\delta_0 >> 1$ the time of formation of the structures in the conductor is $t_0 \sim \ln \delta_0/\delta_0$, i.e., it decreases as δ_0 increases.

Discussion and Conclusions. The present results show that in a metallic conductor under the action of Joule heat sources an inhomogeneity intergrowing across the current lines can appear and thermal and electric structures will be formed in the vicinity of the inhomogeneity. The main parameter determining the conditions for formation of the structures is δ_0. Its physical meaning is the ratio of the characteristic power of the Joule sources to the characteristcic rate of heat removal by heat conduction. The effect of intergrowth of the inhomogeneity can appear in the conductor when $\delta_0 > \delta_{0\,cr}$, i.e., in electrophysical installations of high power. The numerical value of $\delta_{0\,cr}$ depends substantially on the shape of the inhomogeneity. This is already evident from a physical consideration of the extreme cases. Indeed, for manifestation of the intergrowth effect it is necessary that the heat release in the vicinity of the inhomogeneity at points lying on the r axis be substantially higher than that at other points of the medium. But the heat release is proportional to the square of the current density. Therefore, if the inhomogeneity is extended along the z axis in the initial state, the electric current lines flowing around the inhomogeneity are distorted slightly and a high power of the field sources is required in order that intergrowth of the inhomogeneity be induced. In the opposite case, when the inhomogeneity is extended along the z axis at the initial moment, in flowing around the inhomogeneity the electric current lines are distorted quite strongly. Consequently, in the latter case $\delta_{0\,cr}$ will be substantially smaller than in the former case. The value of $\delta_{0\,cr}$ also depends substantially on the characteristic dimension of the inhomogeneity R.

Finally, the formation of stable thermal and electric structures depends substantially on how distinctly the inhomogeneity is singled out against the background of the medium. If the inhomogeneity is such that far from the center the function describing its geometry decreases according to an exponential law with exponents larger than 3/2 along the r axis and larger than 3 along the z axis, then universal structures are formed, the form of which is independent of the form the inhomogeneity.

As follows from the aforesaid, the present results can be used both in scientific research (for example, in constructing models of thermal destruction of metals in strong electromagnetic fields) and in technology (for example, in developing new methods of nondestructive testing of metals).

APPENDIX 1

The steady-state solution of system (1) at $\delta_0 >> 1$ is found by the method described in [1, 2]; therefore, only the main points in the construction of function (9) that are characteristic of the present problem are given here. Assuming that $\Phi_1(r, z) = \Phi_{10}(r, z) + Q_0(\xi, z) + O(\delta_0^{-1})$, $T_1(r, z) = T_{10}(r, z) + \Pi_0(\xi, z) + O(\delta_0^{-1})$, where $\xi = r/\sqrt{\delta_0}$, substituting them into (1), and equating terms with the same powers of δ_0, after some transformations we obtain equations for determining $\Phi_{10}(r, z)$, $T_{10}(r, z)$, $Q_0(\xi, z)$, $\Pi_0(\xi, z)$:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi_{10}}{\partial r}\right) = -\frac{1}{2}\frac{\partial \Omega(r, z)}{\partial z}, \tag{A1}$$

$$T_{10} = -\frac{\partial \Phi_{10}}{\partial z} - \frac{\Omega}{2}, \tag{A2}$$

$$\frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi\frac{\partial Q_0}{\partial \xi}\right) = \frac{1}{2}\frac{\partial^4 Q_0}{\partial z^4}, \quad \Pi_0 = -\frac{\partial Q_0}{\partial z}. \tag{A3}$$

The solution of Eq. (A1) is $\Phi_{10} = -1/2\int_r^\infty dx/x \int_x^\infty y\cdot(\partial\Omega(y, z)/\partial z)dy$ or after integration by parts

$$\Phi_{10}(r, z) = \frac{1}{2}\left[\ln r \int_r^\infty x\frac{\partial \Omega(x, z)}{\partial z}dx - \int_r^\infty x\ln x\frac{\partial \Omega(x, z)}{\partial z}dx\right]. \tag{A4}$$

Then, from Eq. (A2)

$$T_{10}(r, z) = \frac{1}{2}\left[\int_r^\infty x\ln x\frac{\partial^2 \Omega(x, z)}{\partial z^2}dx - \ln r\int_r^\infty x\frac{\partial^2 \Omega(x, z)}{\partial z^2}dx\right] - \frac{\Omega}{2}. \tag{A5}$$

Equations (A3) are independent of $\Omega(r, z)$, and therefore, the form of the functions $Q_0(\xi, z)$ and $\Pi_0(\xi, z)$ coincides with that obtained in [1, 2]:

$$Q_0(\xi, z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \exp(ikz)\, q(k)\, K_0\left(\frac{k^2}{\sqrt{2}}\xi\right)dk, \quad \Pi_0(\xi, z) = -\frac{\partial Q_0}{\partial z}, \tag{A6}$$

where $K_0(x)$ is the modified Bessel function of the second kind. The unknown function $q(k)$ is determined from the obvious condition of finiteness of the solution of problem (1) at any r and z. In this connection it should be noted that from the boundness of $|K_0(x) + \ln(x/2)| < M$ it follows that

$$\left|\int_0^\infty \exp(ikz)\, ka(k)\, [K_0(k^2\xi/\sqrt{2}) + \ln(k^2\xi/\sqrt{8})]\, dk\right| < M\int_0^\infty |ka(k)|\, dk,$$

where $a(k)$ is function (11). But from the absolute integrability of the functions $a_0(z)$ and da_0/dz, convergence of $\int_0^\infty |ka(k)|\, dk$ follows. Therefore

$$\varphi(\xi) = \int_{-\infty}^\infty \exp(ikz)\, ka(k)\, [K_0(k^2\xi/\sqrt{2}) + \ln(k^2\xi/\sqrt{8})]\, dk$$

is a continuous function of ξ and $\lim_{\xi\to 0}\varphi(\xi) = -C\int_{-\infty}^{+\infty} ka(k)\exp(ikz)dk = -C da_0(z)/dz$, where C is the Euler constant, and the value of the limit follows from a power series expansion of the function $K_0(x)$. Consequently, in order to eliminate singularities in solutions (A4)–(A6) arising at $r \to 0$, it must be assumed that $q(k) = a(k)/2$. Then, the distributions of the potential $\Phi_0(0, z)$ and the temperature $T_1(0, z)$ of (10) along the z axis will be finite, and therefore, the functions $\Phi_1(r, z)$ and $T_1(r, z)$ of (9) will be bounded at any r and z.

APPENDIX 2

For simplicity the function $\Omega(r, z)$ is assumed to be even in both r and in z. Then,

$$\int_{-\infty}^{\infty} \exp(ikz)\, ika(k) \ln k dk = -2\,\mathrm{Re} \int_{0}^{\infty} \exp(ikz)\, k^2 \omega(k) \ln k dk . \tag{A7}$$

Asymptotic expressions for such integrals at $z \to \infty$ are found in [6]. As applied to Eq. (A7), this gives

$$\int_{-\infty}^{\infty} \exp(ikz)\, ika(k) \ln k dk \sim -\frac{\sqrt{8\pi}}{z^3} \int_{0}^{\infty} x dx \int_{0}^{\infty} \Omega(x, y)\, dy + o(z^{-3}) . \tag{A8}$$

From condition 4 imposed on the function $\Omega(r, z)$ and from (7) it follows that $\Omega(0, z) = o(z^{-2})$, $\partial^2\Omega/\partial z^2 = o(z^{-4})$, $da_0(z)/dz = o(z^{-4})$, $\int_0^\infty x \ln x \partial^2\Omega(x, z)/\partial z^2 dx = o(z^{-4})$. To find the asymptotic expression for the integral $\int_0^\infty k^2\omega(k)K_0((k^2/\sqrt{2})\xi)dk$ at $r \to \infty$, the function $\omega(k)$ is expanded in a Taylor series. Then

$$\int_0^\infty k^2 \omega(k) K_0\left(\frac{k^2}{\sqrt{2}}\xi\right) dk = \sum_{m=0}^{\infty} \omega^{(m)}(0)/m! \int_0^\infty k^{m+2} K_0\left(\frac{k^2}{\sqrt{2}}\xi\right) dk =$$

$$= \omega(0)\, \Gamma^2\left(\frac{3}{4}\right) \Big/ (2^{3/4} \xi^{3/2}) + o(\xi^{-3/2}) ,$$

where $\Gamma(x)$ is the gamma-function. From Eq. (8), $\Omega(r, 0) = O(r^{-2-\varepsilon})$, $\varepsilon > 0$. If $\partial^2\Omega/\partial z^2 = O(r^{-4-\varepsilon})$,

$$I(r) = \int_r^\infty x \ln\frac{x}{r} \frac{\partial^2 \Omega(x, 0)}{\partial z^2} dx \sim \int_r^\infty \frac{\ln(x/r)}{x^{3+\varepsilon}} dx = \frac{1}{\varepsilon^2} \frac{1}{r^{2+\varepsilon}} , \quad r \to \infty .$$

$$\text{At } \partial^2\Omega/\partial z^2 = O(r^{-2-\varepsilon}),\ I(r) \sim 1/(\varepsilon^2 r^\varepsilon).$$

NOTATION

Dimensionless quantities: r, z, cylindrical coordinates; t, time; σ, electric conductivity; ρ, specific resistance; T_1, temperature; Ω, function describing the geometry of the inhomogeneity; Φ, electric potential; δ_0, parameter of the system. Dimensional quantities: α, temperature coefficient of resistance; χ, thermal diffusivity; γ, density of the material; c, heat capacity; R, characteristic dimension of the inhomogeneity; T_a, ambient temperature; T_0, initial temperature.

REFERENCES

1. L. S. Gerasimov and V. I. Ikryannikov, Élektrichestvo, No. 5, 38-42 (1991).
2. V. I. Ikryannikov, Inzh.-Fiz. Zh., **62**, No. 4, 617-623 (1992).
3. M. A. Abramovitz and I. Stigun (eds.), Reference Book of Elementary Functions [Russian translation], Moscow (1979).
4. A. M. Ter-Krikorov and M. I. Shabunin, Course of Mathematical Analysis [in Russian], Moscow (1988).
5. G. Polya and G. Szego, Aufgaben und Lehrsatze aus der Analysis, Pt. 1, Berlin (1964).
6. J. A. Armstrong and N. Bleistein, SIAM J. Math. Anal. **11**, No. 2, 300-307 (1980).